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TWO-DIMENSIONAL TRANSONIC FLOW PAST AIRFOILS

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SUMMARY

This report concerns the problem of constructing solutions for transonic flows over symmetric airfoils. The aspect of the problem emphasized is, of necessity, not how to form a solution for compressible flow but how to simplify the initial phase of the problem, namely, the mapping of the incompressible flow. In the case of the symmetric Joukowski airfoil without circulation, the mapping is relatively simple, but the coefficients in the power series are difficult to evaluate. As a result, the problem requires simplification. Instead of the exact incompressible flow past the airfoil, an approximate flow is used, which is derived from a combination of source and sink. This flow differs only slightly from the exact one when the thickness is small. By the same method, the flow with circulation is also considered.

After the incompressible-flow functions are approximated in this fashion, the numerical calculation of the corresponding compressible flow, by the hodograph theory, does not present any essential difficulty.

INTRODUCTION

The problem of transonic potential flows has been considered in two previous reports (references 1 and 2). The object was, in the first place, to construct a solution for a closed body, and, secondly, to devise a method by which the flow can be easily calculated. The method of constructing a solution is essentially a method of analytic continuation. That is, a complex potential, which is known as a function of the complex velocity, is represented by a Taylor series about the origin of the hodograph plane and by another series in an annular region. Both series are convergent in their respective domains and agree on the common circle of convergence. In the interior region, a solution for compressible fluid can immediately be obtained by replacing the proper particular integrals in the Taylor series. The new solution does not affect the radius of convergence and, as Mach number tends to zero, reduces to that of the incompressible flow. The solution for the annular region cannot be formed by simply replacing the corresponding particular integrals in

the outside series of the incompressible flow, because the solution so obtained does not agree with the inside series on the circle of convergence and, consequently, does not represent the same function.

The problem is then not so simple as it first appears to be. In order to make both series represent one solution, the idea of retaining the coefficients in both series of the incompressible flow must be abandoned. Instead, both the coefficients as well as the form of the solution are modified to satisfy the conditions on the circle of convergence (reference 2). A different approach to the same problem is that due to Lighthill and Cherry (references 3 and 4). Instead of choosing in advance the form of the outside series, they first transform the Taylor series into a double series, which, on interchanging the order of summation, can be summed to yield a single series. As a result, the series is valid anywhere except at the singularities in the domain considered, and analytic continuation is automatically accomplished. The continued outside series consists of two series; one is derived from the outside series of the incompressible flow and the other is a Taylor series.

The status of the hodograph method is then this: If a complex potential is given in series form, a corresponding solution for compressible fluid can be constructed, the body shape being unknown until the end of the calculation. It is apparent that the fundamental problem of determining the flow pattern about a given body remains unsolved. Moreover, even the transformation of the complex potential often poses many practical difficulties. Nevertheless, the hodograph method is still the only one available by which an exact solution may be sought. For this reason, there is still ground for further exploration along this line.

It is the object of this report to examine a case of more practical interest than that previously considered, in order to discover possible means of simplification. As a first attempt, a symmetric Joukowski airfoil is studied. In this case, if the rule of transforming incompressible to compressible flow is strictly followed, the calculation becomes extremely laborious. However, if a special case of small thickness - which is also of practical interest - is considered, the functions can be simplified to such an extent that numerical calculation is practical. The process is, in a sense, approximate. But as far as the compressible flow is concerned, the merit of a method lies only in whether it yields a good aerodynamic body or not. It has never been demonstrated that an exact incompressible potential function gives a better result. From this point of view, this is just another way of choosing a complex potential.

Finally, it must be emphasized that the present investigation is only the beginning, as far as practical calculations for useful airfoil profiles are concerned. Taking the approximate function as a guide, refinement is not difficult to make by introducing additional terms according to the character of the function. For instance, when circulation is present, the hydrodynamic functions for compressible flow transformed from those of incompressible flow for a symmetric body will give a negatively cambered airfoil. The necessity of applying correction terms is obvious.

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FUNDAMENTAL EQUATIONS AND THEIR PARTICULAR SOLUTIONS

A steady, potential, and isotropic motion of a perfect gas in a plane satisfies a system of equations in the hodograph plane (references 1 and 2):

$$\left.\begin{aligned} q\frac{\partial\varphi}{\partial q} &= -\frac{\rho_o}{\rho}\left(1 - M^2\right)\frac{\partial\psi}{\partial\theta} \\ \frac{\partial\varphi}{\partial\theta} &= \frac{\rho_o}{\rho}\, q\,\frac{\partial\psi}{\partial q} \end{aligned}\right\} \qquad (1)$$

Here $\varphi(q,\theta)$ and $\psi(q,\theta)$ are, respectively, the velocity potential and stream function; q and θ, the magnitude and inclination of the velocity vector, respectively; ρ and M, the density of the fluid and the local Mach number, respectively; and ρ_o, the value of ρ at $q = 0$. (See appendix A for definitions of all symbols.)

This fundamental system, as shown previously (reference 2), can be solved and, in the case of $\psi(q,\theta)$, the particular integrals are

$$\left.\begin{aligned} & q^{\nu}F_{\nu}(\tau)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\nu\theta \\ & q^{-\nu}F_{-\nu}(\tau)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\nu\theta \end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned}&\frac{\partial}{\partial\nu}\left[\tau^{\nu}F_{\nu}(\tau)\cos\nu\theta\right]\\&\frac{\partial}{\partial\nu}\left[\tau^{\nu}F_{\nu}(\tau)\sin\nu\theta\right]\end{aligned}\right\}\tag{3}$$

and

$$\theta \quad \text{and} \quad \int^{\tau}(1-\tau)^{\beta}\,\frac{d\tau}{\tau}\tag{4}$$

and, in the case of $\varphi(q,\theta)$, are

$$\left.\begin{aligned}&(1-\tau)^{-\beta}q^{\nu}F_{\nu}(\tau)\xi_{\nu}(\tau)\begin{Bmatrix}\sin\\-\cos\end{Bmatrix}\nu\theta\\&(1-\tau)^{-\beta}q^{-\nu}F_{-\nu}(\tau)\xi_{-\nu}(\tau)\begin{Bmatrix}\sin\\-\cos\end{Bmatrix}\nu\theta\end{aligned}\right\}\tag{5}$$

$$\left.\begin{aligned}&(1-\tau)^{-\beta}\frac{\partial}{\partial\nu}\left[\tau^{\nu}F_{\nu}(\tau)\xi_{\nu}\sin\nu\theta\right]\\&(1-\tau)^{-\beta}\frac{\partial}{\partial\nu}\left[\tau^{\nu}F_{\nu}(\tau)\xi_{\nu}(\tau)\cos\nu\theta\right]\end{aligned}\right\}\tag{6}$$

$$(1-\tau)^{-\beta}-\frac{1}{2}\int^{\tau}(1-\tau)^{-\beta}\,\frac{d\tau}{\tau} \quad \text{and} \quad \theta\tag{7}$$

All these results are well-known except equations (3) and (6) which can be found in appendix B. Here $F_{\nu}(\tau)$ and $F_{-\nu}(\tau)$ stand, respectively, for the hypergeometric functions $F\left(a_{\nu},b_{\nu};\ \nu+1;\ \tau\right)$ and $F\left(a_{\nu}-\nu,b_{\nu}-\nu;\ 1-\nu;\ \tau\right)$ for nonintegral parameters ν. In case ν

is an integer n, the definition of the second integral is modified. The new function will include a logarithmic term (cf. reference 2) and is denoted by $F_{-n}(\tau)$. The functions $\xi_\nu(\tau)$ and $\xi_{-\nu}(\tau)$ are defined by

$$\left.\begin{aligned} \nu\xi_\nu(\tau) &= 2\tau \frac{d}{d\tau} \log_e \tau^{\nu/2} F_\nu(\tau) \\ \nu\xi_{-\nu}(\tau) &= 2\tau \frac{d}{d\tau} \log_e \tau^{-\nu/2} F_{-\nu}(\tau) \end{aligned}\right\} \tag{8}$$

and the constants a_ν and b_ν are defined by

$$\left.\begin{aligned} a_\nu + b_\nu &= \nu - \beta \\ a_\nu b_\nu &= -\frac{1}{2}\beta\nu(\nu + 1) \end{aligned} \qquad \beta = \frac{1}{\gamma - 1}\right\} \tag{9}$$

and the variable τ is

$$\tau = \frac{1}{2\beta}\frac{q^2}{c_o^2} \tag{10}$$

where γ is the ratio of specific heats of the gas and c_o, the speed of sound at $q = 0$.

Furthermore, $F_\nu(\tau)$ is an analytic function of both variables τ and ν. For fixed $\tau < \frac{\gamma - 1}{\gamma + 1}$, it has simple poles at $\nu = -n$ and, accordingly, possesses an expansion of the form (references 3 and 4):

$$F_\nu(\tau) = f(\tau)T^\nu - \sum_{m=1}^{\infty} \frac{c_m}{\nu + m} \tau^m T(\tau)^{m+\nu} F_m(\tau) \tag{11}$$

where

$$\left.\begin{aligned}
f(\tau) &= \frac{(1-\tau)^{\gamma_1^2/4}}{\left(1-\gamma_1^2\tau\right)^{1/4}} \\
T(\tau) &= \frac{2}{\left(1-\gamma_1\right)^{\gamma_1}} \frac{\left[\gamma_1(1-\tau)^{1/2} + \left(1-\gamma_1^2\tau\right)^{1/2}\right]^{\gamma_1}}{(1-\tau)^{1/2} + \left(1-\gamma_1^2\tau\right)^{1/2}} \qquad \gamma_1 = \sqrt{\frac{\gamma+1}{\gamma-1}} \\
c_m &= \frac{\Gamma\left(a_m\right)\Gamma\left(m+1-b_m\right)}{\Gamma\left(a_m-m\right)\Gamma\left(1-b_m\right)m!(m-1)!}
\end{aligned}\right\} \tag{12}$$

This expansion is valid for all values of ν except negative integers. When $\nu = -n$ the limiting process gives

$$F_{-n}(\tau) = f(\tau)T^{-n}(\tau) - \sum_{m=1}^{\infty}{}' \frac{c_m}{-n+m} \tau^m T^{m-n}(\tau) F_m(\tau) - c_n\left[\tau^n F_n(\tau) \log_e T + \frac{\partial}{\partial n}\left(\tau^n F_n\right)\right] \tag{13}$$

where the prime indicates that the term $m = n$ is to be excluded.

From these particular solutions one can construct a solution for any boundary in the hodograph plane. Once the functions $\varphi(q,\theta)$ and $\psi(q,\theta)$ are known, the flow pattern in the physical plane is given by integrating

$$\left.\begin{aligned}
\frac{\partial x}{\partial \theta} &= \frac{1}{q}\left(\cos\theta \frac{\partial \varphi}{\partial \theta} - \frac{\rho_o}{\rho} \sin\theta \frac{\partial \psi}{\partial \theta}\right) \\
\frac{\partial y}{\partial \theta} &= \frac{1}{q}\left(\sin\theta \frac{\partial \varphi}{\partial \theta} + \frac{\rho_o}{\rho} \cos\theta \frac{\partial \psi}{\partial \theta}\right)
\end{aligned}\right\} \tag{14}$$

and

$$\left.\begin{aligned}\frac{\partial x}{\partial q} &= \frac{1}{q}\left(\cos\theta\,\frac{\partial\varphi}{\partial q} - \frac{\rho_o}{\rho}\sin\theta\,\frac{\partial\psi}{\partial q}\right)\\ \frac{\partial y}{\partial q} &= \frac{1}{q}\left(\sin\theta\,\frac{\partial\varphi}{\partial q} + \frac{\rho_o}{\rho}\cos\theta\,\frac{\partial\psi}{\partial q}\right)\end{aligned}\right\} \tag{15}$$

where x and y are the coordinates of a point in the physical plane.

MAPPING OF AN INCOMPRESSIBLE FLOW

The potential flow of an incompressible fluid past a symmetric Joukowski airfoil without circulation is determined by a complex function ${}_{o}W(z)$:

$${}_{o}W(z) = \zeta + \epsilon + \frac{(1+\epsilon)^2}{\zeta+\epsilon} \qquad z = \zeta + \frac{1}{\zeta} \tag{16}$$

where

$$z = x + iy$$

$$\zeta = \xi + i\eta$$

Here all the physical quantities are made dimensionless by dividing, respectively, every length and velocity by the radius of the generating circle and the free-stream velocity. Then ϵ is a purely real number. The complex velocity w for such a flow is

$$w = \frac{\zeta^2(\zeta + 1 + 2\epsilon)}{(\zeta+1)(\zeta+\epsilon)^2} \tag{17}$$

This equation expresses the relationship between w and ζ. It can also be interpreted as a transformation function from the ζ-plane to the w-plane. In other words, if a flow field in the physical plane is given, its image in the w-plane is immediately known by means of equation (17). Since the present problem is formulated in the w-plane instead of the z-plane, the flow field in the z-plane can be calculated from the inverse transformation $\zeta(w)$, which is given by solving

$$\zeta^3 + (1 + 2\epsilon)\zeta^2 - \frac{\epsilon(\epsilon + 2)w}{1 - w}\zeta - \frac{\epsilon^2 w}{1 - w} = 0 \tag{18}$$

The three solutions of this equation are:

$$\left.\begin{aligned}
\zeta_1(w) &= P(w) + Q(w) - \frac{1 + 2\epsilon}{3} \\
\zeta_2(w) &= -\frac{1}{2}\left[P(w) + Q(w)\right] - \frac{i\sqrt{3}}{2}\left[P(w) - Q(w)\right] - \frac{1 + 2\epsilon}{3} \\
\zeta_3(w) &= -\frac{1}{2}\left[P(w) + Q(w)\right] + \frac{i\sqrt{3}}{2}\left[P(w) - Q(w)\right] - \frac{1 + 2\epsilon}{3}
\end{aligned}\right\} \tag{19}$$

where

$$\left.\begin{aligned}
P(w) &= -\frac{1}{3}\left\{(1 - \epsilon)^3 + \frac{9\epsilon(1 + \epsilon + \epsilon^2)}{1 - w} - \right. \\
&\qquad \left. \frac{i3^{3/2}\epsilon(1 + 2\epsilon)^{3/2}w^{1/2}}{(1 - w)^{3/2}}\left[1 - \frac{(1 - \epsilon)^3(1 + \epsilon)}{(1 + 2\epsilon)^3}w\right]\right\} \\
Q(w) &= -\frac{1}{3}\left\{(1 - \epsilon)^3 + \frac{9\epsilon(1 + \epsilon + \epsilon^2)}{1 - w} + \right. \\
&\qquad \left. \frac{i3^{3/2}\epsilon(1 + 2\epsilon)^{3/2}w^{1/2}}{(1 - w)^{3/2}}\left[1 - \frac{(1 - \epsilon)^3(1 + \epsilon)}{(1 + 2\epsilon)^3}w\right]\right\}
\end{aligned}\right\} \tag{20}$$

This indicates that to a neighborhood in the w-plane there correspond, in general, three distinct neighborhoods in the ζ-plane or the z-plane (see fig. 1). The exceptional points where two of the three solutions become equal are given by $dw/d\zeta = 0$. This yields three points $\zeta = 0$, $\zeta = -\frac{1 + 2\epsilon}{2 + \epsilon}$, and $\zeta = \infty$ about which two of three

solutions join. Furthermore, it remains to investigate how the axis of reals, that is, $v = 0$, is mapped in the ζ-plane. By separating into real and imaginary parts, the velocity component v in terms of ξ and η is:

$$v = -\frac{2\epsilon\eta\left(\xi^2 + \epsilon\xi + \eta^2\right)\left[(2 + \epsilon)\xi + 1 + 2\epsilon\right]}{(\zeta\bar{\zeta} + 2\xi + 1)\left(\zeta\bar{\zeta} + 2\epsilon\xi + \epsilon^2\right)^2}$$

Then $v = 0$ corresponds to

$$\eta = 0$$

$$(2 + \epsilon)\xi + 1 + 2\epsilon = 0$$

$$\left(\xi + \frac{\epsilon}{2}\right)^2 + \eta^2 = \frac{\epsilon^2}{4}$$

In other words, the axis of reals of the w-plane corresponds to: (a) The axis of reals of the ζ-plane, (b) a straight line through one branch point parallel to the η-axis, and (c) a circle. Moreover, while u increases from zero to one along $v = 0$, $\zeta_1(w)$ decreases from the forward stagnation point to negative infinity, $\zeta_2(w)$ increases from zero to positive infinity, and $\zeta_3(w)$ decreases to a negative number. Evidently, branches I and II join smoothly at the line $\xi = -\dfrac{1 + 2\epsilon}{2 + \epsilon}$ and branches II and III meet along the circle $\left(\xi + \dfrac{\epsilon}{2}\right)^2 + \eta^2 = \dfrac{\epsilon^2}{4}$. Branch III then lies entirely inside the circle of radius $\epsilon/2$. Hence, the flow over a body with a sharp trailing edge, complicated as it is, remains essentially the same as that over the body which is symmetric about two axes with two stagnation points. Finally, by substituting equations (19) in equation (16), the corresponding branches of the complex potential are ${}_0W_1(w)$, ${}_0W_2(w)$, and ${}_0W_3(w)$.

When circulation is present, having the magnitude required by the condition of finite trailing-edge velocity at an angle of attack α, the complex potential is

$${}_0W(z) = (\zeta + \epsilon)e^{-i\alpha} + \frac{(1 + \epsilon)^2}{\zeta + \epsilon}e^{i\alpha} + 2i(1 + \epsilon)\sin\alpha\log_e(\zeta + \epsilon) \quad (21)$$

with $z = \zeta + \frac{1}{\zeta}$; and the complex velocity is

$$w = \frac{\zeta^2\left[(\zeta + \epsilon)e^{-i\alpha} + (1 + \epsilon)e^{i\alpha}\right]}{(\zeta + 1)(\zeta + \epsilon)^2} \tag{22}$$

The inverse transformation function $\zeta(w)$ in this case consists in the solution of

$$\left(e^{-i\alpha} - w\right)\zeta^3 + \left[\epsilon e^{-i\alpha} + (1 + \epsilon)e^{i\alpha} - (1 + 2\epsilon)w\right]\zeta^2 - \epsilon(2 + \epsilon)w\zeta - \epsilon^2 w = 0 \tag{23}$$

The solution of this equation can, of course, be given, but the expressions are so complex as to make any practical manipulation difficult. To bring out the complicated character of the Reimann surface it is sufficient to determine the branch points and branch lines. Now the branch points are $\zeta = 0$, $\zeta = \infty$ and

$$(1 + \epsilon)\left(1 - e^{2\alpha i}\right)\zeta^2 + \epsilon\left[3 + \epsilon + (1 + \epsilon)e^{2\alpha i}\right]\zeta + 2\epsilon\left[\epsilon + (1 + \epsilon)e^{2\alpha i}\right] = 0$$

of which two are inside the body; and the wind axis in the hodograph plane, that is, the axis in the direction of the wind, corresponds, even for small values of α, to curves in the ζ-plane:

$$\alpha(1 + \epsilon)(\xi + 1 + 2\epsilon)\eta^4 + \epsilon\left[(2 + \epsilon)\xi + 1 + 2\epsilon\right]\eta^3 + \alpha(1 + \epsilon)\left[5\xi^3 + 6(1 + \epsilon)\xi^2 - \left(3 - 2\epsilon - \epsilon^2\right)\xi - (1 + \epsilon)^2\right]\eta^2 + \epsilon\xi(\xi + \epsilon)\left[(2 + \epsilon)\xi + 1 + 2\epsilon\right]\eta + \alpha(1 + \epsilon)\xi^2(\xi + 1)(\xi + \epsilon)^2 = 0$$

This equation is of degree four in η and five in ξ. It would be interesting to trace this equation for given values of ϵ and α. This was not done for the sole reason that the exact problem is too difficult, if not impossible, to handle. It should be remembered that the present procedure of constructing a solution of a compressible flow depends primarily on the knowledge of an incompressible-flow solution in the form of power series. Even in the case of zero circulation, as

can be seen from equations (20), the series expansion of the complex potential yields sets of coefficients which are, at least, double infinite series (cf. appendix C). One can easily see that the initial task alone is a formidable one, not to mention the later computation of the compressible-flow solution. This amount of labor is particularly unjustifiable in this problem because, even if one starts with an exact solution, he may still end up with a body very different from what he expected because of the distortion due to compressibility. For this reason deviation from the formal procedure is recommended wherever it is convenient. In this report, the case of small thickness and small angle of attack is studied first.

Before proceeding further, it is mentioned that the Joukowski airfoil is characterized by having a cusp at the trailing edge. Although the velocity is finite there, the acceleration is infinite and, consequently, the analyticity of the mapping function breaks down; namely, $d_oW/dw = 0$. In order to preserve this property in the case of compressible flow, the stream function $\psi(q,\theta)$ must satisfy

$$\left.\begin{aligned} \frac{\partial\psi}{\partial q} &= 0 \\ \frac{\partial\psi}{\partial\theta} &= 0 \end{aligned}\right\} \tag{24}$$

In the case of symmetric flow, the first is satisfied identically. The second serves to determine the speed at the trailing edge.

APPLICATION TO THIN SYMMETRIC AIRFOILS

Consider the case of a thin symmetric airfoil present in a uniform flow without circulation, namely, $\alpha = 0$. Since the thickness ratio of a Joukowski airfoil is directly proportional to the parameter ϵ, for thin airfoils ϵ must be small. Suppose that the airfoil is so thin that terms higher than ϵ are negligible. Equation (18) is then simplified to

$$\zeta^2 + (1 + 2\epsilon)\zeta - \frac{2\epsilon w}{1 - w} = 0 \tag{25}$$

of which the solutions are:

$$\left.\begin{aligned}\zeta_1(w) &= -\frac{1}{2}(1 + 2\epsilon)\left(1 + \sqrt{\frac{1 - \epsilon_1^2 w}{1 - w}}\right)\\ \zeta_2(w) &= -\frac{1}{2}(1 + 2\epsilon)\left(1 - \sqrt{\frac{1 - \epsilon_1^2 w}{1 - w}}\right)\end{aligned}\right\} \tag{26}$$

where $\epsilon_1 = \dfrac{1 - 2\epsilon}{1 + 2\epsilon}$, being of the order of magnitude unity.

By making such an approximation it is seen that the third solution disappears. This is explained by the fact that the third branch lies wholly inside a circle of area $\frac{1}{4}\pi\epsilon^2$ which, according to the present approximation, is zero; therefore $\zeta_3(w) = 0$. The branch points are now located at $\zeta = -\frac{1}{2}(1 + 2\epsilon)$ and $\zeta = \infty$, connected by the branch line (fig. 2) $2\xi + 1 + 2\epsilon = 0$.

The problem is indeed simple. The question, however, is: What flow is represented or, in other words, how is the complex potential modified? By using the simplified transformation function, a straightforward integration gives

$$_0W(z) = \zeta + \epsilon + \frac{1 + 2\epsilon}{2\epsilon} \log_e \left(1 + \frac{2\epsilon}{\zeta}\right) \tag{27}$$

The flow in the ζ-plane is then that produced by the superposition of a sink and a source at, respectively, $\zeta = 0$ and $\zeta = -2\epsilon$ in a uniform flow. It is very easy to see that, if ϵ is small, the vanishing of the stream function corresponds closely to a circle with radius $1 + \epsilon$ and center at $\zeta = -\epsilon$. Hence for small values of ϵ the curve in the z-plane will not differ much from a Joukowski airfoil. By substituting ζ_1 and ζ_2 from equations (26) in equation (27) the two branches of the complex potential are

$$\left.\begin{aligned}
{}_0W_1(w) &= -\frac{1}{2}\left[1 + (1 + 2\epsilon)\sqrt{\frac{1 - \epsilon_1{}^2 w}{1 - w}}\right] + \\
&\qquad \frac{1 + 2\epsilon}{2\epsilon}\log_e\left[1 - \frac{4\epsilon}{1 + 2\epsilon}\Big/\left(\frac{1 - \epsilon_1{}^2 w}{1 - w}\right)^{1/2} + 1\right] \\
{}_0W_2(w) &= -\frac{1}{2}\left[1 - (1 + 2\epsilon)\sqrt{\frac{1 - \epsilon_1{}^2 w}{1 - w}}\right] + \\
&\qquad \frac{1 + 2\epsilon}{2\epsilon}\log_e\left[1 - \frac{4\epsilon}{1 + 2\epsilon}\Big/ -\left(\frac{1 - \epsilon_1{}^2 w}{1 - w}\right)^{1/2} + 1\right]
\end{aligned}\right\} \tag{28}$$

By taking $\epsilon = 0.08$, the hodograph $\psi_0(q,\theta) = 0$ from equations (28) and that of a symmetric Joukowski airfoil are compared in figure 3. It is seen that, except in the neighborhood of the singularity $w = 1$, the agreement is quite close even for a thickness ratio of about 10 percent.

For the case where a weak circulation is present, by the same method of approximation, equation (23) simplifies to

$$\zeta^2 + \left(1 + 2\epsilon + \frac{2\alpha i}{1 - w}\right)\zeta - \frac{2\epsilon w}{1 - w} = 0 \tag{29}$$

Here α is assumed to be the same order of magnitude as ϵ, and therefore the product of ϵ and α is also dropped. The solutions of this equation are

$$\left.\begin{aligned}
\zeta_1(w) &= -\frac{1 + 2\epsilon}{2}\left(1 + \sqrt{\frac{1 - \epsilon_1{}^2 w}{1 - w}}\right) + \alpha i\left[-\frac{1}{1 - w} - \frac{1}{(1 - w)^{1/2}\left(1 - \epsilon_1{}^2 w\right)^{1/2}}\right] \\
\zeta_2(w) &= -\frac{1 + 2\epsilon}{2}\left(1 - \sqrt{\frac{1 - \epsilon_1{}^2 w}{1 - w}}\right) + \alpha i\left[-\frac{1}{1 - w} + \frac{1}{(1 - w)^{1/2}\left(1 - \epsilon_1{}^2 w\right)^{1/2}}\right]
\end{aligned}\right\} \tag{30}$$

the corresponding complex potential is

$$ {}_0W(w) = (1 - \alpha i)\zeta + \epsilon + \frac{1 + 2\epsilon + \alpha i}{2\epsilon} \log_e \left(1 + \frac{2\epsilon}{\zeta}\right) + 2\alpha i \log_e (\zeta + 2\epsilon) \tag{31} $$

The body in the ζ-plane is again a circle, provided both ϵ and α are small. A substitution of ζ_1 and ζ_2 from equations (30) in equation (31) gives, respectively, ${}_0W_1(w)$ and ${}_0W_2(w)$.

It should be noted that, in the previous case when circulation was present, it altered radically the nature of the singularity of the flow. Now by comparing equations (30) with equations (26) one can see that circulation has introduced essentially no complication and hence the problem can be similarly handled as in the case where no circulation exists.

EXPANSIONS OF ${}_0W(w)$

When $\alpha = 0$.- For thin airfoils, as seen from the previous section, the domain of interest has two branch points $\zeta = -\frac{1}{2}(1 + 2\epsilon)$ and $\zeta = \infty$; the corresponding points in the w-plane are $w = 1$ and $w = \frac{1}{{\epsilon_1}^2}$, at each of which ${}_0W(w)$ becomes singular. As a result, two different expansions are required to represent each branch of ${}_0W(w)$. In the case of ${}_0W_1(w)$, the origin is a regular point; for ${}_0W_2(w)$ it is a logarithmic singularity.

Consider ${}_0W_1(w)$ for the case $\alpha = 0$ first: About the forward stagnation point $|\zeta| \neq 0$ and ${}_0W_1(w)$ can be simplified in the form:

$$ {}_0W_1(w) = \zeta_1 + \epsilon + \frac{1 + 2\epsilon}{\zeta_1} - \frac{\epsilon(1 + 2\epsilon)}{{\zeta_1}^2} + 0\left(\epsilon^2\right) \tag{32} $$

By means of the relation:

$$ \zeta_1\zeta_2 = -\frac{2\epsilon w}{1 - w} $$

${}_0W_1(w)$ can be reduced to two elements $(1 - w)^{-1/2}\left(1 - \epsilon_1{}^2w\right)^{-1/2}$ and $(1 - w)^{1/2}\left(1 - \epsilon_1{}^2w\right)^{1/2}$ which possess the following expansions in the respective domains:

$$\left.\begin{aligned}(1 - w)^{-1/2}\left(1 - \epsilon_1{}^2w\right)^{-1/2} &= \sum_0^\infty S_n{}^{(i)}w^n \\ (1 - w)^{1/2}\left(1 - \epsilon_1{}^2w\right)^{1/2} &= \sum_0^\infty {}_1S_n{}^{(i)}w^n\end{aligned}\quad |w| < 1 \right\} \tag{33}$$

where

$$\left.\begin{aligned}S_n{}^{(i)} &= (-1)^n \sum_{m=0}^{n} \binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n - m}\epsilon_1{}^{2(n-m)} \\ {}_1S_n{}^{(i)} &= (-1)^n \sum_{m=0}^{n} \binom{\frac{1}{2}}{m}\binom{\frac{1}{2}}{n - m}\epsilon_1{}^{2(n-m)}\end{aligned}\right\} \tag{34}$$

and

$$\left.\begin{aligned}(1 - w)^{-1/2}\left(1 - \epsilon_1{}^2w\right)^{-1/2} &= \sum_0^\infty S_n{}^{(o)}\epsilon_1{}^{2n}w^n + \\ &\quad \sum_1^\infty S_n{}^{(o)}w^{-n} \\ (1 - w)^{1/2}\left(1 - \epsilon_1{}^2w\right)^{1/2} &= \sum_0^\infty {}_1S_n{}^{(o)}\epsilon_1{}^{2n}w^n + \\ &\quad \sum_1^\infty {}_1S_n{}^{(o)}w^{-n}\end{aligned}\quad 1 < |w| < \frac{1}{\epsilon_1{}^2}\right\} \tag{35}$$

where

$$\left.\begin{aligned} S_n^{(o)} &= (-1)^n \sum_0^\infty \binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{m+n}\epsilon_1^{2m} \\ {}_1S_n^{(o)} &= (-1)^n \sum_0^\infty \binom{\frac{1}{2}}{m}\binom{\frac{1}{2}}{m+n}\epsilon_1^{2m} \end{aligned}\right\} \tag{36}$$

A simple reduction then yields

$${}_oW_1(w) = -A_o - \sum_2^\infty A_n w^n + O\left(\epsilon^2\right) \qquad |w| < 1 \tag{37}$$

where

$$A_n = \frac{1+2\epsilon}{2}\left(S_n^{(i)} - \epsilon_1^2 S_{n-1}^{(i)}\right) + \frac{(1+2\epsilon)^2}{4\epsilon}\left[S_{n+1}^{(i)} - \left(1+\epsilon_1^2\right)S_n^{(i)} + \epsilon_1^2 S_{n-1}^{(i)}\right] + \frac{(1+2\epsilon)^3}{8\epsilon}\left[-S_{n+2}^{(i)} + \left(2+\epsilon^2\right)S_{n+1}^{(i)} - 2\epsilon_1^2 S_n^{(i)} - S_n^{(i)} + \epsilon_1^2 S_{n-1}^{(i)}\right] \tag{38}$$

Here A_2 is zero, as is easily shown from equation (27); in the above expansion it would have been of the order of ϵ^2 and hence is included in the symbol $O\left(\epsilon^2\right)$. The velocity potential and stream function are, neglecting $O\left(\epsilon^2\right)$,

$$\left.\begin{aligned} {}_o\varphi_1 &= -A_o - \sum_2^\infty A_n q^n \cos n\theta \\ {}_o\psi_1 &= \sum_2^\infty A_n q^n \sin n\theta \end{aligned}\right\} \quad q < 1 \tag{39}$$

If the expansions (35) are introduced, in the annulus region one will have

$$ {}_{o}W_{1}(w) = B_{-2}w^{-2} + B_{-1}w^{-1} + B_{o} + i\sum_{0}^{\infty}\left(B_{\nu}w^{\nu} + C_{\nu}w^{-\nu}\right) \qquad 1 < |w| < \frac{1}{\epsilon_{1}^{2}} \tag{40} $$

where

$$ \left.\begin{aligned} B_{o} &= -\frac{(1 + 2\epsilon)(3 + 8\epsilon)}{8\epsilon} \\ B_{-1} &= \frac{(1 + 2\epsilon)^{2}}{2\epsilon} \\ B_{-2} &= -\frac{(1 + 2\epsilon)^{3}}{8\epsilon} \\ B_{\nu} &= \frac{1 + 2\epsilon}{2}\left(S_{n+1}(o) - S_{n}(o)\right) - \frac{(1 + 2\epsilon)^{2}}{4\epsilon}\,{}_{1}S_{n+1}(o) - \\ &\quad \frac{(1 + 2\epsilon)^{3}}{8\epsilon}\left({}_{1}S_{n+1}(o) - {}_{1}S_{n+2}(o)\right) \\ C_{\nu} &= \frac{1 + 2\epsilon}{2}\left(S_{n}(o) - \epsilon_{1}^{2}S_{n+1}(o)\right) - \frac{(1 + 2\epsilon)^{2}}{4\epsilon}\,{}_{1}S_{n}(o) - \\ &\quad \frac{(1 + 2\epsilon)^{3}}{8\epsilon}\left({}_{1}S_{n}(o) - {}_{1}S_{n-1}(o)\right) \\ C_{1/2} &= \frac{1 + 2\epsilon}{2}\left(S_{o}(o) - \epsilon_{1}^{2}S_{1}(o)\right) - \frac{(1 + 2\epsilon)^{2}}{4\epsilon}\,{}_{1}S_{o}(o) - \\ &\quad \frac{(1 + 2\epsilon)^{3}}{8\epsilon}\left({}_{1}S_{o}(o) - F_{1}^{2}\,{}_{1}S_{1}(o)\right) \\ \nu &= n + \frac{1}{2} \end{aligned}\right\} \tag{41} $$

Separating into real and imaginary parts, ${}_o\varphi_1$ and ${}_o\psi_1$ are, approximately,

$$\left.\begin{aligned} {}_o\varphi_1 &= B_{-2}q^{-2}\cos 2\theta + B_{-1}q^{-1}\cos\theta + B_o + \sum_0^\infty\left(B_\nu q^\nu - C_\nu q^{-\nu}\right)\sin\nu\theta \\ {}_o\psi_1 &= B_{-2}q^{-2}\sin 2\theta + B_{-1}q^{-1}\sin\theta + \sum_0^\infty\left(B_\nu q^\nu + C_\nu q^{-\nu}\right)\cos\nu\theta \end{aligned}\right\} \tag{42}$$

for $1 < q < \dfrac{1}{\epsilon_1{}^2}$.

In the case of ${}_oW_2(w)$, as $w = 0$ is not a regular point, a Taylor expansion does not exist. It can be expanded, however, in the following form:

$${}_oW_2(w) = \sum_0^2 A_{-n}w^{-n} + \sum_2^\infty A_nw^n + O\left(\epsilon^2\right) \qquad |w| < 1 \tag{43}$$

where A_n is defined in equation (38) while A_{-o}, A_{-1}, and A_{-2} are

$$\left.\begin{aligned} A_{-o} &= -\frac{3(1 + 2\epsilon)}{4\epsilon} \\ A_{-1} &= \frac{(1 + 2\epsilon)^2}{\epsilon} \\ A_{-2} &= -\frac{(1 + 2\epsilon)^3}{4\epsilon} \end{aligned}\right\} \tag{44}$$

On the other hand, in the region $1 < |w| < \epsilon_1{}^{-2}$ the expansions are

$${}_oW_2(w) = \sum_0^2 B_{-n}w^{-n} - i\sum_0^\infty\left(B_\nu w^\nu + C_\nu w^{-\nu}\right) + O\left(\epsilon^2\right) \tag{45}$$

Here the constants B_{-2}, B_{-1}, B_o, B_ν, and C_ν are defined in equations (41). It is clear that by circling once about $w = 1$ branch I goes smoothly over into branch II, as is easily seen from equations (40) and (45). Similarly, the velocity potential and stream function are, omitting $O(\epsilon^2)$,

$$\left.\begin{aligned} {}_o\varphi_2 &= \sum_0^2 A_{-n}q^{-n}\cos n\theta + \sum_2^\infty A_n q^n \cos n\theta \\ {}_o\psi_2 &= \sum_1^2 A_{-n}q^{-n}\sin n\theta + \sum_2^\infty A_n q^n \sin n\theta \end{aligned}\right\} \quad 0 < q < 1 \qquad (46)$$

and for $1 < q < \epsilon^{-2}$

$$\left.\begin{aligned} {}_o\varphi_2 &= \sum_0^2 B_{-n}q^{-n}\cos n\theta + \sum_0^\infty \left(B_\nu q^\nu - C_\nu q^{-\nu}\right)\sin \nu\theta \\ {}_o\psi_2 &= \sum_1^2 B_{-n}q^{-n}\sin n\theta - \sum_0^\infty \left(B_\nu q^\nu + C_\nu q^{-\nu}\right)\cos \nu\theta \end{aligned}\right\} \qquad (47)$$

When $\alpha \neq 0$.- When α is finite but small in the case of a thin airfoil, equations (30) show that

$$\left.\begin{aligned} \zeta_1 &= \zeta_1^{(o)} + \alpha\zeta_1^{(1)} \\ \zeta_2 &= \zeta_2^{(o)} + \alpha\zeta_2^{(1)} \end{aligned}\right\} \qquad (48)$$

where $\zeta_1^{(o)}$ and $\zeta_2^{(o)}$ are the transformation functions for $\alpha = 0$ and $\zeta_1^{(1)}$ and $\zeta_2^{(1)}$ are correction terms due to circulation. Therefore, they are defined as follows:

$$\left.\begin{aligned}
\zeta_1^{(o)} &= -\frac{1 + 2\epsilon}{2}\left(1 + \sqrt{\frac{1 - \epsilon_1^2 w}{1 - w}}\right)\\
\zeta_2^{(o)} &= -\frac{1 + 2\epsilon}{2}\left(1 - \sqrt{\frac{1 - \epsilon^2 w}{1 - w}}\right)\\
-i\zeta_1^{(1)} &= -\frac{1}{1 - w} - \frac{1}{(1 - w)^{1/2}\left(1 - \epsilon_1^2 w\right)^{1/2}}\\
-i\zeta_2^{(1)} &= -\frac{1}{1 - w} + \frac{1}{(1 - w)^{1/2}\left(1 - \epsilon_1^2 w\right)^{1/2}}
\end{aligned}\right\} \qquad (49)$$

Similarly, the complex potential can be written as

$$\left.\begin{aligned}
{}_oW_1 &= {}_oW_1^{(o)} + \alpha\, {}_oW_1^{(1)}\\
{}_oW_2 &= {}_oW_2^{(o)} + \alpha\, {}_oW_2^{(1)}
\end{aligned}\right\} \qquad (50)$$

where ${}_oW_1^{(o)}$ and ${}_oW_2^{(o)}$ are defined in equations (28) and ${}_oW_1^{(1)}$ and ${}_oW_2^{(1)}$ are found to be

$$\left.\begin{aligned}
-i\,{}_oW_1^{(1)} &= \zeta_1^{(1)} - \zeta_1^{(o)} + \frac{1 + 4\epsilon}{2\epsilon}\log_e\left(\zeta_1^{(o)} + 2\epsilon\right) - \\
&\qquad \frac{1}{2\epsilon}\log_e \zeta_1^{(o)} - \frac{(1 + 2\epsilon)\zeta_1^{(1)}}{\zeta_1^{(o)}\left(\zeta_1^{(o)} + 2\epsilon\right)}\\
-iW_2^{(1)} &= \zeta_2^{(1)} - \zeta_2^{(o)} + \frac{1 + 4\epsilon}{2\epsilon}\log_e\left(\zeta_1^{(o)} + 2\epsilon\right) - \\
&\qquad \frac{1}{2\epsilon}\log_e \zeta_2^{(o)} - \frac{(1 + 2\epsilon)\zeta_2^{(1)}}{\zeta_2^{(o)}\left(\zeta_2^{(o)} + 2\epsilon\right)}
\end{aligned}\right\} \qquad (51)$$

which are valid if $\zeta^{(o)} \neq 0$.

To expand ${}_0W(w)^{(1)}$, besides the fundamental forms, expansions (33) and (35), it is required also to consider

$$\log_e \left[(1 - 2\epsilon)(1 - w)^{1/2} + (1 + 2\epsilon)\left(1 - \epsilon_1{}^2 w\right)^{1/2}\right]$$

and

$$\log_e \left[(1 - w)^{1/2} + \left(1 - \epsilon_1{}^2 w\right)^{1/2}\right]$$

For the first expression let

$$F(w) = \frac{d}{dw} \log_e \left[(1 - 2\epsilon)(1 - w)^{1/2} + (1 + 2\epsilon)\left(1 - \epsilon_1{}^2 w\right)^{1/2}\right]$$

By direct differentiation one obtains

$$F(w) = -\frac{\epsilon_1}{2} (1 - w)^{-1/2}\left(1 - \epsilon_1{}^2 w\right)^{-1/2}$$

By making use of equations (33) and (35), a term-by-term integration gives

$$\log_e \left[(1 - 2\epsilon)(1 - w)^{1/2} + (1 + 2\epsilon)\left(1 - \epsilon_1{}^2 w\right)^{1/2}\right] =$$

$$-\frac{\epsilon_1}{2} \sum_1^\infty \frac{S_{n-1}{}^{(i)}}{n} w^n + \log_e 2 \qquad |w| < 1 \qquad (52a)$$

and, aside from a constant,

$$\log_e \left[(1 - 2\epsilon)(1 - w)^{1/2} + (1 + 2\epsilon)\left(1 - \epsilon_1{}^2 w\right)^{1/2}\right] =$$

$$\frac{i\epsilon_1}{2} \sum_0^\infty \frac{1}{\nu}\left(-S_{n+1}{}^{(o)} w^{-\nu} + S_n{}^{(o)} \epsilon_1{}^{2n} w^{\nu}\right) \qquad (52b)$$

A similar method yields

$$\log_e \left[(1 - w)^{1/2} + \left(1 - \epsilon_1{}^2 w\right)^{1/2}\right] =$$

$$-\frac{1}{2}\sum_{1}^{\infty} \frac{S_n{}^{(i)}}{n} w^n + \log_e 2 \qquad |w| < 1 \qquad (53a)$$

and

$$\log_e \left[(1 - w)^{1/2} + \left(1 - \epsilon_1{}^2 w\right)^{1/2}\right] =$$

$$\frac{1}{2}\left[\log_e w + i\sum_{0}^{\infty} \frac{1}{\nu}\left(S_{n+1}{}^{(o)}\epsilon_1{}^{2n+2}w^{\nu} - S_n{}^{(o)}w^{-1}\right)\right] \quad 1 < |w| < \epsilon_1{}^{-2} \qquad (53b)$$

Furthermore, $\dfrac{\zeta_1{}^{(1)}}{\zeta_1{}^{(o)}\left(\zeta_1{}^{(o)} + \epsilon\right)}$ can be written in an expansible form:

$$-\frac{1}{2\epsilon}\left(1 + \epsilon_1\sqrt{\frac{1 - w}{1 - \epsilon_1{}^2 w}}\right)$$

Collecting all the individual expansions, one finds

$$-i\ {}_oW_1{}^{(1)} = 2\pi i + \sum_{2}^{\infty} A_n{}^{(1)} w^n + O\left(\epsilon^2\right) \qquad |w| < 1 \qquad (54)$$

where

$$A_n{}^{(1)} = -\left(1 + S_n{}^{(i)}\right) + \frac{1 + 2\epsilon}{2}\left(S_n{}^{(i)} - \epsilon_1{}^2 S_{n-1}{}^{(i)}\right) +$$

$$\frac{1 + 4\epsilon}{4\epsilon}\frac{1}{n}\left(1 - \epsilon_1{}^2 S_{n-1}{}^{(i)}\right) - \frac{1}{4\epsilon}\frac{1}{n}\left(1 - S_n{}^{(i)}\right) +$$

$$\frac{1 - 2\epsilon}{2\epsilon}\left(S_{n-1}{}^{(i)} - S_n{}^{(i)}\right) \qquad (55)$$

Here $A_1^{(1)}$ is of the order of ϵ^2 and therefore neglected. The real and imaginary parts of the correction term of ${}_oW(w)$ are, to the order of approximation considered,

$$\left.\begin{aligned} {}_o\varphi_1^{(1)} &= -2\pi + \sum_2^\infty A_n^{(1)}q^n \sin n\theta \\ {}_o\psi_1^{(1)} &= \sum_2^\infty A_n^{(1)}q^n \cos n\theta \end{aligned} \qquad q < 1 \right\} \tag{56}$$

In the region $1 < |w| < \epsilon_1^{-2}$, with the exception of a constant

$$-i\ {}_oW_1^{(1)} = -\frac{1 + 4\epsilon}{4\epsilon} \log_e we^{\pi i} + \sum_1^\infty \left(1 + \frac{1}{n}\right)w^{-n} +$$

$$\sum_0^\infty \left(B_\nu^{(1)}w^\nu + C_\nu^{(1)}w^{-\nu}\right) + O\left(\epsilon^2\right) \tag{57}$$

where

$$\left.\begin{aligned} B_\nu^{(1)} = &\left[S_{n+1}^{(o)}\epsilon_1^2 - \frac{1 + 2\epsilon}{2}\left(S_{n+1}^{(o)} - S_n^{(o)}\right)\epsilon_1^2 + \frac{1 + 4\epsilon}{4\epsilon\nu} S_n^{(o)} - \right. \\ &\left. \frac{1}{4\epsilon\nu} S_{n+1}^{(o)}\epsilon_1^2 + \frac{1 + 2\epsilon}{2\epsilon}\left(S_{n+1}^{(o)}\epsilon_1^2 - S_n^{(o)}\right)\epsilon_1\right]\epsilon_1^{2n} \\ C_\nu^{(1)} = &\ S_n^{(o)} - \frac{1 + 2\epsilon}{2}\left(S_n^{(o)} - \epsilon_1^2 S_{n+1}^{(o)}\right) - \frac{(1 + 4\epsilon)\epsilon_1}{4\epsilon\nu} S_{n+1}^{(o)} + \\ &\frac{1}{4\epsilon\nu} S_n^{(o)} + \frac{1 + 2\epsilon}{2\epsilon}\left(S_{n+1}^{(o)} - S_n^{(o)}\right)\epsilon_1 \end{aligned}\right\} \tag{58}$$

The corresponding functions ${}_{o}\varphi_1{}^{(1)}$ and ${}_{o}\psi_1{}^{(1)}$ are

$$\left.\begin{aligned}
{}_{o}\varphi_1{}^{(1)} &= \frac{1 + 4\epsilon}{4\epsilon}(\pi - \theta) - \sum_{1}^{\infty}\left(1 + \frac{1}{n}\right)q^{-n}\sin n\theta + \\
&\quad \sum_{0}^{\infty}\left(B_\nu{}^{(1)}q^\nu - C_\nu{}^{(1)}q^{-\nu}\right)\sin \nu\theta + \text{Constant} \\
{}_{o}\psi_1{}^{(1)} &= -\frac{1 + 4\epsilon}{4\epsilon}\log_e q + \sum_{1}^{\infty}\left(1 + \frac{1}{n}\right)q^{-n}\cos n\theta + \\
&\quad \sum_{0}^{\infty}\left(B_\nu{}^{(1)}q^\nu + C_\nu{}^{(1)}q^{-\nu}\right)\cos \nu\theta + \text{Constant}
\end{aligned}\right\} \tag{59}$$

Similarly, for the second branch the expansion is

$$-i\,{}_{o}W_2{}^{(1)} = C_o{}^{(1)} - \frac{1}{2\epsilon}\pi i - \frac{1}{2\epsilon}\log_e w + \sum_{2}^{\infty} C_n{}^{(1)}w^n + O\left(\epsilon^2\right) \qquad |w| < 1 \tag{60}$$

where

$$\left.\begin{aligned}
C_o{}^{(1)} &= \frac{1 + 4\epsilon}{4\epsilon}\log_e \epsilon - \frac{1}{2\epsilon}\log_e \frac{1 + 2\epsilon}{2} + 2 \\
C_n{}^{(1)} &= -\left(1 - S_n{}^{(i)}\right) - \frac{1 + 2\epsilon}{2}\left(S_n{}^{(i)} - \epsilon_1{}^2 S_{n-1}{}^{(i)}\right) + \\
&\quad \frac{1 + 4\epsilon}{4\epsilon n}\left(1 + \epsilon_1 S_{n-1}{}^{(i)}\right) - \frac{1}{4\epsilon n}\left(1 + S_n{}^{(i)}\right) + \\
&\quad \frac{1 + 2\epsilon}{2\epsilon}\left(S_n{}^{(i)} - S_{n-1}{}^{(i)}\right)\epsilon_1
\end{aligned}\right\} \tag{61}$$

For the region $1 < |w| < \epsilon_1^{-2}$, it is

$$-i\ {}_oW_2^{(1)} = -\frac{1}{2\epsilon}\log_e w - \log_e we^{\pi i} + \sum_1^{\infty}\left(1 + \frac{1}{n}\right)w^{-n} -$$

$$\sum_0^{\infty}\left(B_\nu^{(1)}w^{\nu} + C_\nu w^{-\nu}\right) + O\left(\epsilon^2\right) \tag{62}$$

where a constant of order unity is again left out. The corresponding real and imaginary parts are:

$$\left.\begin{aligned} {}_o\varphi_2^{(1)} &= \frac{1}{2\epsilon}(\pi - \theta) + \sum_2^{\infty} C_n^{(1)}q^n \sin n\theta \\ {}_o\psi_2^{(1)} &= C_o^{(1)} - \frac{1}{2\epsilon}\log_e q + \sum_2^{\infty} C_n^{(1)}q^n \cos n\theta \end{aligned}\right\} \quad q < 1 \tag{63}$$

and, aside from a constant, in $1 < q < \epsilon_1^{-2}$,

$$\left.\begin{aligned} {}_o\varphi_2^{(1)} &= -\frac{1}{2\epsilon}\theta + (\pi - \theta) - \sum_1^{\infty}\left(1 + \frac{1}{n}\right)q^{-n}\sin n\theta - \\ &\quad \sum_0^{\infty}\left(B_\nu^{(1)}q^{\nu} - C_\nu^{(1)}q^{-\nu}\right)\sin \nu\theta \\ {}_o\psi_2^{(1)} &= -\frac{1 + 4\epsilon}{4\epsilon}\log_e q + \sum_1^{\infty}\left(1 + \frac{1}{n}\right)q^{-n}\cos n\theta - \\ &\quad \sum_0^{\infty}\left(B_\nu^{(1)}q^{\nu} + C_\nu^{(1)}q^{-\nu}\right)\cos \nu\theta \end{aligned}\right\} \tag{64}$$

CONSTRUCTION OF A SOLUTION FOR A SYMMETRIC AIRFOIL

The complex potential ${}_0W$ for a flow over a thin Joukowski airfoil has now been transformed and expanded in power series of w for both the case of zero circulation and of a weak circulation. Knowing the form of expansion of ${}_0W$, a solution relating to a flow of incompressible fluid about a certain body can be obtained immediately by replacing each term of the series by the proper particular integral as given in the section "Fundamental Equations and Their Particular Solutions." For instance, the complex potential in the case of zero circulation is

$$
{}_0W_1(w) = -\sum_{0}^{\infty} A_n w^n \qquad |w| < 1
$$

The complex potential $W_1(w,\tau)$ relating to a compressible flow is, accordingly,

$$
W_1(w,\tau) = -\sum_{0}^{\infty} A_n F_n^{(r)}(\tau) w^n + \sum_{m=2}^{\infty} h_m c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m e^{im\theta} -
$$

$$
A_2 c_2 \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n e^{in\theta}\right]_{n=2} \qquad |w| < 1 \tag{65}
$$

of which the imaginary part gives the stream function, namely,

$$
\psi_1(q,\theta) = \mathrm{Im}\left[W_1(w,\tau)\right] \tag{66}
$$

Here the definition of $F_n^{(r)}(\tau)$ may be either $F_n(\tau)/F_n(\tau_1)$ (references 1 and 2) or $F_n(\tau)/T_1(\tau_1)$ (references 3 and 4), where $\tau = \tau_1$ when $q = 1$. It follows from equation (66) that the stream function is

$$
\psi_{1-}(q,\theta) = \sum_{2}^{\infty} A_n q^n F_n^{(r)}(\tau) \sin n\theta + \sum_{2}^{\infty} h_m c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m \sin m\theta -
$$

$$
A_2 c_2 \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n \sin n\theta\right]_{n=2} \qquad q < 1 \tag{67}
$$

From equation (5) the velocity potential is

$$\varphi_{1-}(q,\theta) = A_0 - (1 - \tau)^{-\beta} \sum_{2}^{\infty} A_n q^n F_n^{(r)}(\tau)\xi_n(\tau) \cos n\theta + \sum_{2}^{\infty} h_m c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m \xi_m \cos m\theta - A_2 c_2 \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n \xi_n \cos n\theta\right]_{n=2}$$

$$q < 1 \qquad (68)$$

Both series (equations (67) and (68)) converge only within the unit circle $q = 1$. In order to have a function to cover the whole field of flow, both series must be continued across the circle of convergence. There have been two alternative methods proposed to effect this continuation. The first method (references 1 and 2) is to have the solution corresponding to the flow in the upper z-plane arranged in one w-plane. After the series, for example, equations (39), (42), and (46), are modified, they are required to be continuous with continuous normal derivatives on the circle $q = 1$ (fig. 4). On account of compressibility, the modified solution will be discontinuous in normal derivative and hence W_{1-} and W_1 can no longer represent the same function. To correct this, one set of coefficients is left free and an extra series is added to W_1. These two sets of coefficients can then be determined by the condition at $q = 1$ (reference 2). The second procedure (references 3 and 4) is to proceed from a Taylor expansion, equation (67), say. Expressing the function $F_n(\tau)$ in the form of equation (11) gives rise to a convergent double series which, when the order of summation is changed, is transformed into a form being convergent even for values of q greater than unity. Analytical continuation is thus accomplished. In the following discussion, the latter procedure is adopted for simplicity.

At this point, the convention for numbering the branch of ${}_0W(w)$ is altered. Instead of dividing the z-plane fore and aft, it will be more convenient to divide the plane into upper and lower halves. Then ${}_0W_1(w)$ indicates the upper half and ${}_0W_2(w)$, the lower. The parts corresponding to the regions $q < 1$ fore, $1 < q < \epsilon_1^{-2}$, and $0 < q < 1$ aft are, respectively, ${}_0W_{1-}$, ${}_0W_1$, and ${}_0W_{1+}$.

By the second method, replacing $F_n(\tau)$ in equation (65) by equation (11) and, for evident reasons, adopting the definition $F_n^{(r)}(\tau) = F_n(\tau)/T_1(\tau_1)$, equation (65) can be written as

$$W_1(w,\tau) = -f(\tau) \sum_{0}^{\infty} A_n (tw)^n + \sum_{n=0}^{\infty} A_n (tw)^n \sum_{m=1}^{\infty} \frac{C_m}{n + m} \tau^m T^m F_m(\tau) + \sum_{2}^{\infty} h_m c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m e^{im\theta} - A_2 c_2 \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n e^{in\theta}\right]_{n=2}$$

where $t = T/T_1$. The first series can be summed and, by changing the order of summation, the second series can be transformed into a single series. Thus

$$W_1(w,\tau) = f(\tau)\ {}_0W_1(\tilde{w}) - \sum_{m=1}^{\infty} C_m\left(\sqrt{\tau_1 T_1}\right)^m \psi_m(\tau) e^{im\theta} \int_0^{\tilde{w}} w^{m-1}\ {}_0W_1\ dw +$$

$$\sum_2^{\infty} h_m c_m\left(\sqrt{\tau_1 T_1}\right)^m \psi_m e^{im\theta} - A_2 c_2 \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1 T_1}\right)^n \psi_n e^{in\theta}\right]_{n=2} \tag{69}$$

where $\tilde{w} = tw$ and $\psi_m(\tau) = \tau^{m/2} F_m(\tau)$. The first term corresponds to that resulting from the asymptotic summation (references 1 and 2) and, in the special case $\gamma = -1$, reduces to the Kármán-Tsien approximation. The integral in the second term is bounded and the series can be shown to be convergent for $q > 1$ (references 3 and 4).

By taking a path starting from the forward stagnation point to an arbitrary point in the annular region, namely,

$$\int_0^{\tilde{w}} w^{m-1}\ {}_0W_1(w)\ dw = \int_0^{\tilde{w}_1} w^{m-1}\ {}_0W_1(w)\ dw + \int_{\tilde{w}_1}^{\tilde{w}} w^{m-1}\ {}_0W_1(w)\ dw$$

where $\tilde{w}_1$ is a fixed point in the annular region, a simple transformation leads to

$$W_1 = \sum_0^{2}{}' B_{-n}F_{-n}^{(r)}(\tau)w^{-n} + i\sum_0^{\infty}\left[B_\nu F_\nu^{(r)}(\tau)w^{\nu} + C_\nu F_{-\nu}^{(r)}(\tau)w^{-\nu}\right] +$$

$$\left(-A_2c_2 + B_{-2}c_2\right)\frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1 T_1}\right)^n \psi_n e^{in\theta}\right]_{n=2} \tag{70}$$

by replacing ${}_oW_1$ by the proper expansion under the integral sign, where

$$h_m = \int_0^{\tilde{w}_1} w^{m-1} {}_oW_1 \, dw - \sum_0^{2}{}' \frac{B_{-n}\tilde{w}_1^{-n}}{-n + m} - i \sum_0^{\infty} \left(\frac{B_\nu \tilde{w}_1^{\nu+w}}{\nu + m} + \frac{C_\nu \tilde{w}_1^{-\nu+w}}{-\nu + m} \right) - B_{-2} \log_e \tilde{w}_1 \tag{71}$$

The integral in equation (71) can be evaluated in the ζ-plane; namely,

$$\int_0^{\tilde{w}_1} w^{m-1} {}_oW_1 \, dw = \int_{\zeta_o}^{\tilde{\zeta}_1} w^{m-1} {}_oW_1 \frac{dw}{d\zeta} d\zeta$$

The fact that $\frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1}T_1\right)^n \psi_n e^{in\theta}\right]$ is a solution is shown in appendix B. Equation (70), evidently, is the continuation of equation (65). Proceeding further to a fixed point $\tilde{w}_2$ inside the unit circle but below the real axis, by describing a loop surrounding the point $w = 1$, the following proper transformation is arrived at:

$$W_1 = \sum_2^{\infty} h_m c_m \left(\sqrt{\tau_1}T_1\right)^m \psi_m e^{im\theta} + \sum_1^{2} A_{-2}F_{-n}{}^{(r)}(\tau)\, w^{-n} + \sum_0^{\infty} A_n F_n{}^{(r)}(\tau) w^n \tag{72}$$

where

$$h_m(\tilde{\zeta}_2) = \int_{\tilde{\zeta}_o}^{\tilde{\zeta}_2} w^{m-1} {}_oW_1 \frac{dw}{d\zeta} d\zeta - \sum_1^{2}{}' \frac{A_n \tilde{w}_2^{-n+m}}{-n + m} - \sum_0^{\infty} \frac{A_n' \tilde{w}_2^{n+m}}{n + m} - A_{-2} \log_e \tilde{w}_2 \tag{73}$$

The path is quite arbitrary; in order to preserve the property of symmetry of the function it is so chosen that the h_m's are real. Equations (65), (70), and (72) together represent the first branch of the Riemann surface covering the entire upper z-plane. Separating into real and imaginary parts, the stream function is

$$\psi_1(q,\theta) = \sum_0^\infty \left[B_\nu q^\nu F_\nu^{(r)}(\tau) + C_\nu q^{-\nu} F_{-\nu}^{(r)}(\tau)\right] \cos \nu\theta +$$

$$\sum_0^2 B_{-n} q^{-n} F_{-n}^{(r)}(\tau) \sin n\theta - \Big(A_2 c_2 -$$

$$B_{-2} c_2\Big) \frac{\partial}{\partial n} \left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n \sin n\theta\right]_{n=2} \qquad 1 < q < \epsilon_1^{-2} \tag{74}$$

and

$$\psi_{1+}(q,\theta) = \sum_2^\infty h_m\left(\tilde{\zeta}_2\right) c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m(\tau) \sin m\theta -$$

$$\sum_2^\infty A_n q^n F_n^{(r)}(\tau) \sin n\theta +$$

$$\sum_1^2 A_{-n} q^{-n} F_{-n}^{(r)}(\tau) \sin n\theta \qquad q < 1 \tag{75}$$

The velocity potential corresponding to equations (74) and (75) is

$$\varphi_1(q,\theta) = (1 - \tau)^{-\beta} \left\{ \sum_0^2 B_{-n} q^{-n} F_{-n}^{(r)} \xi_{-n} \cos n\theta - \right.$$

$$\sum_0^\infty \left[B_\nu q^\nu F_\nu^{(r)}(\tau)\xi_\nu + C_\nu q^{-\nu} F_{-\nu}^{(r)}(\tau)\xi_{-\nu} \right] \sin \nu\theta +$$

$$\left. \left(A_2 c_2 - B_{-2} c_2\right) \frac{\partial}{\partial n} \left[\left(\sqrt{\tau_1 T_1}\right)^n \psi_n \xi_n \cos n\theta \right]_{n=2} \right\} \qquad 1 < q < \epsilon_1^{-2} \tag{76}$$

and

$$\varphi_{1+}(q,\theta) = (1 - \tau)^{-\beta} \left\{ - \sum_2^\infty h_m\left(\tilde{\zeta}_2\right) c_m \left(\sqrt{\tau_1 T_1}\right)^m \psi_m(\tau) \xi_m \cos m\theta - \right.$$

$$\sum_1^2 A_{-n} q^{-n} F_{-n}^{(r)}(\tau) \xi_{-n} \cos n\theta +$$

$$\left. \sum_2^\infty A_n q^n F_n^{(r)}(\tau) \xi_n \cos n\theta \right\} \qquad 0 < q < 1 \tag{77}$$

By taking a path from the forward stagnation point but in the reverse direction, the second branch of $W(w,\tau)$ can be joined. Now the solution ${}_0W(w)$ to be considered in the integral appearing in equation (69) will be

$$\left.\begin{aligned}
{}_0W_{2-} &= -\sum_0^{\infty} A_n w^n && |w| < 1 \\
{}_0W_2 &= \sum_0^{2} B_{-n} w^{-n} - i\sum_0^{\infty}\left(B_\nu w^\nu + C_\nu w^{-\nu}\right) && 1 < |w| < \epsilon_1^{-2} \\
{}_0W_{2+} &= \sum_1^{2} A_{-n} w^{-n} + \sum_0^{\infty} A_n w^{+n} && |w| < 1
\end{aligned}\right\} \tag{78}$$

Complete symmetry with respect to the u-axis makes it unnecessary to repeat the process.

In the case of the flow with circulation, the complex potential consists of two parts as shown in equations (50). The one which corresponds to zero circulation is given in equations (37), (40), and (43). The functions for the compressible flow are then given from equations (67) to (77). The part which takes into account the effect of a weak circulation is

$$\left.\begin{aligned}
-i\,{}_0W_{1-}^{(1)} &= \sum_0^{\infty} A_n^{(1)} w^n && |w| < 1 \\
-i\,{}_0W_1^{(1)} &= B_o - \frac{1}{4\epsilon} we^{\pi i} + \sum_1^{\infty}\left(1 + \frac{1}{n}\right)w^{-n} + \sum_0^{\infty}\left(B_\nu^{(1)} w^\nu + \right. \\
&\qquad \left. C_\nu^{(1)} w^{-\nu}\right) - \log_e we^{\pi i} \\
-i\,{}_0W_{1+}^{(1)} &= -\frac{1}{2\epsilon}\log_e w + \sum_0^{\infty} C_n^{(1)} w^n && 0 < |w| < 1
\end{aligned}\right\} \tag{79}$$

By the convention adopted, ${}_oW_{1-}{}^{(1)}$, ${}_oW_1{}^{(1)}$, and ${}_oW_{1+}{}^{(1)}$ form the first branch of ${}_oW^{(1)}$ and correspond to the whole upper z-plane. The same method yields a solution for a compressible flow:

$$
\left.\begin{aligned}
-iW_{1-}{}^{(1)} &= \sum_0^\infty A_n{}^{(1)}F_n{}^{(r)}(\tau)w^n + \sum_2^\infty h_m{}^{(1)}c_m\left(\sqrt{\tau_1}T_1\right)^m\psi_m e^{im\theta} - \\
&\quad \sum_2^\infty \left(1 + \frac{1}{n}\right)c_n \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1}T_1\right)^n\psi_n e^{in\theta}\right] \\
-iW_1{}^{(1)} &= B_o - \frac{1}{4\epsilon}(\pi - \theta)i - \frac{1}{4\epsilon}\int_{\tau_1}^{\tau}(1 - \tau)^\beta \frac{d\tau}{\tau} + \\
&\quad \sum_1^\infty \left(1 + \frac{1}{n}\right)F_{-n}{}^{(r)}w^{-n} + \sum_0^\infty \left[B_\nu{}^{(1)}F_\nu{}^{(r)}w^\nu + \right. \\
&\quad \left. C_\nu{}^{(1)}F_{-\nu}{}^{(r)}w^{-\nu}\right] - \left[(\pi - \theta)i + \int_{\tau_1}^{\tau}(1 - \tau)^\beta \frac{d\tau}{\tau}\right] \\
-iW_{1+}{}^{(1)} &= \sum_2^\infty h_m{}^{(2)}c_m\left(\sqrt{\tau_1}T_1\right)^m\psi_m(\tau)e^{im\theta} - \\
&\quad \sum_2^\infty \left(1 + \frac{1}{n}\right)c_n \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1}T_1\right)^n\psi_n e^{in\theta}\right] - \\
&\quad \frac{1}{2\epsilon}\left[-\theta i + \int_{\tau_1}^{\tau}(1 - \tau)^\beta \frac{d\tau}{\tau}\right] + \sum_0^\infty C_n{}^{(1)}F_n{}^{(r)}w^n
\end{aligned}\right\} \quad (80)
$$

The real and imaginary parts are

$$
\left.
\begin{aligned}
\psi_{1-}^{(1)} &= \sum_{2}^{\infty} A_n^{(1)} q^n F_n^{(r)}(\tau) \cos n\theta + \\
&\quad \sum_{2}^{\infty} h_m^{(1)} c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m \cos m\theta - \\
&\quad \sum_{2}^{n} \left(1 + \frac{1}{n}\right) c_n \frac{\partial}{\partial n} \left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n \cos n\theta\right] \qquad q < 1 \\
\psi_1^{(1)} &= -\frac{1 + 4\epsilon}{4\epsilon} \int_{\tau_1}^{\tau} (1 - \tau)^{\beta} \frac{d\tau}{\tau} - \sum_{1}^{\infty} \left(1 + \frac{1}{n}\right) q^{-n} F_{-n}^{(r)}(\tau) \cos n\theta + \\
&\quad \sum_{0}^{\infty} \left[B_\nu^{(1)} q^{\nu} F_\nu^{(r)}(\tau) + C_\nu^{(1)} q^{-\nu} F_{-\nu}^{(r)}(\tau)\right] \cos \nu\theta + \\
&\quad \text{Constant} \qquad 1 < q < \epsilon_1^{-2} \\
\psi_{1+}^{(1)} &= -\frac{1}{2\epsilon} \int_{\tau_1}^{\tau} (1 - \tau)^{\beta} \frac{d\tau}{\tau} + \sum_{2}^{\infty} h_m^{(2)} c_m \left(\sqrt{\tau_1} T_1\right)^m \psi_m \cos m\theta - \\
&\quad \sum_{2}^{\infty} \left(1 + \frac{1}{n}\right) c_n \frac{\partial}{\partial n} \left[\left(\sqrt{\tau_1} T_1\right)^n \psi_n \cos n\theta\right] + \\
&\quad \sum_{0}^{\infty} C_n^{(1)} q^n F_n^{(r)}(\tau) \cos n\theta \qquad 0 < q < 1
\end{aligned}
\right\} \quad (81)
$$

and

$$
\left.
\begin{aligned}
\varphi_{1-}^{(1)} &= (1-\tau)^{-\beta}\left\{\sum_{2}^{\infty} A_n^{(1)} q^n F_n^{(r)}(\tau)\xi_n \sin n\theta + \right. \\
&\quad \sum_{2}^{\infty} h_m^{(1)} c_m\left(\sqrt{\tau_1}T_1\right)^m \psi_m \xi_m \sin m\theta - \\
&\quad \left. \sum_{2}^{\infty}\left(1+\frac{1}{n}\right)c_n \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1}T_1\right)^n \psi_n \xi_n \sin n\theta\right]\right\} \\
\varphi_{1}^{(1)} &= (1-\tau)^{-\beta}\left\{\sum_{1}^{\infty}\left(1+\frac{1}{n}\right) q^{-n} F_{-n}^{(r)}(\tau)\xi_{-n} \sin n\theta + \right. \\
&\quad \left. \sum_{0}^{\infty}\left[B_n^{(1)} q^{\nu} F_{\nu}^{(r)}(\tau)\xi_{\nu} + C_{\nu}^{(1)} q^{-\nu} F_{-\nu}^{(r)}(\tau)\xi_{-\nu}\right] \sin \nu\theta\right\} + \\
&\quad \text{Constant} - \frac{1+4\epsilon}{4\epsilon}(\pi - \theta) \quad , \qquad 1 < q < \epsilon_1^{-2} \\
\varphi_{1+}^{(1)} &= (1-\tau)^{-\beta}\left[-\sum_{2}^{\infty} h_m^{(2)} c_m\left(\sqrt{\tau_1}T_1\right)^m \psi_m \sin m\theta - \right. \\
&\quad \sum_{2}^{\infty}\left(1+\frac{1}{n}\right)c_n \frac{\partial}{\partial n}\left[\left(\sqrt{\tau_1}T_1\right)^n \psi_n \xi_n \sin n\theta\right] + \\
&\quad \left. \sum_{0}^{\infty} C_n^{(1)} q^n F_n^{(r)}(\tau)\xi_n \sin n\theta - \frac{1}{2\epsilon}(\pi - \theta)\right] \qquad q < 1
\end{aligned}
\right\} \quad (82)
$$

In a similar fashion, the second branch can also be connected by taking a path in the reverse direction and introducing successively the functions:

$$
\left.
\begin{aligned}
-i\ {}_0W_{2-}^{(1)} &= \sum_0^\infty A_n^{(1)} w^n && |w| < 1 \\
-i\ {}_0W_2^{(1)} &= -\frac{1 + 4\epsilon}{4\epsilon} \log_e we^{\pi i} + \sum_1^\infty \left(1 + \frac{1}{n}\right) w^{-n} - \\
&\quad \sum_0^\infty \left(B_\nu^{(1)} w^\nu + C_\nu^{(1)} w^{-\nu}\right) + \text{Constant} && 1 < |w| < \epsilon^{-2} \\
-i\ {}_0W_{2+}^{(1)} &= -\frac{1}{2\epsilon} \log_e w + \sum_0^\infty C_n^{(1)} w^n && |w| < 1
\end{aligned}
\right\} \quad (83)
$$

However, it should be added that, since the functions $W_1^{(1)}$ and $W_2^{(1)}$ do not merely differ by a sign, the $h_m^{(1)}$'s in $W_{2+}^{(1)}$ and $W_{1+}^{(1)}$ will not generally be equal. As a result, the two branches may not be joined with each other across the branch line $y = 0$, $x_2 < x < \infty$. In order to avoid this difficulty, the lower limit in the integral, for instance, equation (73), should be adjusted to make them equal.

Having the stream function and the velocity potential so chosen, the flow pattern in the physical plane can be calculated by integrating equations (14) and (15) to give the coordinate functions $x(q,\theta)$ and $y(q,\theta)$. When the flow is free of circulation, the stream function is antisymmetrical about the x-axis. By taking $x = x_1$, $y = 0$ as the stagnation point and $x = x_2$, $y = 0$ as the trailing edge, the constants of integration can be determined on the unit circle for $\theta > 0$. By symmetry, the reversal of the sign of θ gives the lower half of the flow.

When circulation is present, the symmetry property is destroyed and hence both branches have to be calculated separately. Since

both $W^{(o)}$ and $W^{(1)}$ are properly joined functions, the determination of integration constants on the unit circle for both branches would be sure to give a pair of continuous coordinate functions.

Cornell University
Ithaca, N. Y., August 10, 1948

APPENDIX A

SYMBOLS

q	magnitude of velocity vector
θ	inclination of velocity vector
ρ_o	value of ρ at $q = 0$
ρ	density of fluid
M	local Mach number
ψ	stream function
φ	velocity potential
$F_\nu(\tau), F_{-\nu}(\tau)$	hypergeometric functions
ν	nonintegral parameter
m,n	integers
$\tau = \dfrac{1}{2\beta} \dfrac{q^2}{c_o^2}$	
a_ν, b_ν	parameters of hypergeometric functions
$\beta = \dfrac{1}{\gamma - 1}$	
$\xi_\nu(\tau), \xi_{-\nu}(\tau)$	functions defined by equations (8)
γ	ratio of specific heats of gas
c_o	speed of sound at $q = 0$
$T(\tau)$	defined by equations (12)
c_m	defined by equations (12)
$f(\tau)$	defined by equations (12)

$\gamma_1 = \sqrt{\frac{\gamma + 1}{\gamma - 1}}$

x,y	coordinates of point in physical plane
${}_oW(z)$	complex potential
$z = x + iy$	
ϵ	geometric parameter of body
w	complex velocity defined by equation (17)
$P(w),Q(w)$	defined by equations (20)
u,v	velocity components
$\zeta(w)$	transformation function defined by equation (18)
$\zeta_1(w),\zeta_2(w),\zeta_3(w)$	solutions of equation (18) given by equations (19)
${}_oW_1(w),{}_oW_2(w),{}_oW_3(w)$	branches of complex potential
α	angle of attack

$\epsilon_1 = \frac{1 - 2\epsilon}{1 + 2\epsilon}$

${}_oW(w)$	complex potential for incompressible flow in w
ψ_o	stream function for incompressible flow
φ_o	velocity potential for incompressible flow
$S_n^{(i)},{}_1S_n^{(i)}$	defined by equations (34)
$S_n^{(o)},{}_1S_n^{(o)}$	defined by equations (36)
A_n	coefficient defined by equation (38)
${}_o\varphi_1,{}_o\psi_1$	defined by equations (39)

$B_{-2}, B_{-1}, B_\nu, C_\nu$	defined by equations (41)
${}_o\varphi_2, {}_o\psi_2$	defined by equations (46) and (47)
$\zeta_1^{(o)}, \zeta_2^{(o)}$	transformation functions
$\zeta_1^{(1)}, \zeta_2^{(1)}$	correction terms due to circulation
${}_oW_1^{(o)}, {}_oW_2^{(o)}$	defined by equations (28)
${}_oW_1^{(1)}, {}_oW_2^{(1)}$	defined by equations (51)
$B_\nu^{(1)}, C_\nu^{(1)}$	defined by equations (58)
${}_o\varphi_1^{(1)}, {}_o\psi_1^{(1)}$	defined by equations (59)
$C_o^{(1)}, C_n^{(1)}$	defined by equations (61)
${}_o\varphi_2^{(1)}, {}_o\psi_2^{(1)}$	defined by equations (63) and (64)
$W_1(w,\tau)$	defined by equation (65)
$F_n^{(r)}(\tau) = F_n(\tau)/T_1(\tau_1)$	
τ_1	value of τ at $q = 1$
T_1	value of T at $q = 1$
$\psi_{1-}, \psi_1, \psi_{1+}$	branches of stream function
$\varphi_{1-}, \varphi_1, \varphi_{1+}$	branches of velocity potential
$t = T/T_1$	

$_0W_1(\tilde{w})$ defined by equation (69)

$\tilde{w} = tw$

$\psi_m(\tau) = \tau^{m/2}F_m(\tau)$

h_m defined by equation (71)

$_0W_{1-}^{(1)}, {}_0W_1^{(1)}, {}_0W_{1+}^{(1)}$ correction terms due to circulation

$\psi_{1-}^{(1)}, \psi_1^{(1)}, \psi_{1+}^{(1)}$ defined by equations (81)

$\varphi_{1-}^{(1)}, \varphi_1^{(1)}, \varphi_{1+}^{(1)}$ defined by equations (82)

$_0W_{2-}^{(1)}, {}_0W_2^{(1)}, {}_0W_{2+}^{(1)}$ defined by equations (83)

APPENDIX B

PARTICULAR SOLUTIONS OF FUNDAMENTAL EQUATIONS OF MOTION

If $\psi_m e^{in\theta}$ is a particular integral of differential equations (1), $\frac{\partial}{\partial n}\left(\psi_n e^{in\theta}\right)$ is also a solution. For the differential equation then becomes

$$\frac{d}{dq}\left(\frac{\rho_o}{\rho} q \frac{d\psi_n}{dq}\right) - n^2 \frac{\rho_o}{\rho}\left(1 - M^2\right)\psi_n = 0$$

$$\frac{d}{dq}\left[\frac{\rho_o}{\rho} q \frac{d}{dq}\left(\frac{\partial\psi_n}{\partial n}\right)\right] - n^2 \frac{\rho_o}{\rho q}\left(1 - M^2\right)\frac{\partial\psi_n}{\partial n} = 2n \frac{\rho_o}{\rho}\left(1 - M^2\right)\psi_n$$

As $\psi_n(q)$ satisfies the first equation, the second equation determines $\partial\psi_n/\partial n$. Now

$$d\varphi = -\frac{\rho_o}{\rho}\left(1 - M^2\right)\frac{\partial\psi}{\partial\theta} dq + \frac{\rho_o}{\rho} q \frac{\partial\psi}{\partial q} d\theta$$

By substituting $\frac{\partial}{\partial n}\left[\psi_n(q)e^{in\theta}\right]$ for ψ, it is easy to show

$$d\varphi = \frac{1}{n}\frac{\partial}{\partial q}\left(-i \frac{\rho_o q}{\rho}\frac{\partial^2\psi_n}{\partial n\partial q} + \theta \frac{\rho_o q}{\rho}\frac{\partial\psi_n}{\partial q} + \frac{i}{n}\frac{\rho_o q}{\rho}\frac{\partial\psi_n}{\partial q}\right)e^{in\theta} dq +$$

$$\frac{1}{n}\frac{\partial}{\partial\theta}\left(-i \frac{\rho_o q}{\rho}\frac{\partial^2\psi_n}{\partial n\partial q} + \theta \frac{\rho_o q}{\rho}\frac{\partial\psi_n}{\partial q} + \frac{i}{n}\frac{\rho_o q}{\rho}\frac{\partial\psi_n}{\partial q}\right)e^{in\theta} d\theta$$

Therefore

$$\varphi = -i \frac{\rho_o q}{\rho}\frac{\partial}{\partial n}\left(\frac{1}{n}\frac{\partial\psi_n}{\partial q} e^{in\theta}\right)$$

APPENDIX C

SERIES EXPANSION OF COMPLEX POTENTIAL

The Taylor expansion of the complex potential of a flow over a symmetric Joukowski airfoil is, for $0 < |w| < 1$,

$$ {}_oW_1 = \sum_0^\infty A_n w^n $$

$$ {}_oW_2 = -\frac{1}{2} \sum_0^\infty A_n w^n - \frac{\sqrt{3}}{2} \sum_0^\infty B_\nu w^\nu $$

$$ {}_oW_3 = -\frac{1}{2} \sum_0^\infty A_n w^n + \frac{\sqrt{3}}{2} \sum_0^\infty B_\nu w^\nu $$

where

$$ A_n = C_n + (1 + \epsilon)^2 D_n $$

$$ B_\nu = C_\nu + (1 + \epsilon)^2 D_\nu $$

$$ C_n = -\frac{2(1 + 2\epsilon)}{3} \sum_{j=0}^{\infty} \sum_{k=0}^{j/2} \sum_{l=0}^{k} \binom{\frac{1}{3}}{j} \binom{j}{2k} \binom{k}{l} \binom{-j - k}{n + k - j - l} (-1)^{n-j} 3^{2j-k} \frac{\epsilon^j \left(1 + \epsilon + \epsilon^2\right)^{j-2k} (1 - \epsilon)^{3l} (1 + \epsilon)^l}{(1 + 2\epsilon)^{3(j-k+l)}} $$

$$ C_\nu = -\frac{2(1 + 2\epsilon)}{3} \sum_{j=0}^{\infty} \sum_{k=0}^{j/2} \sum_{l=0}^{\infty} \binom{\frac{1}{3}}{j} \binom{j}{2k + 1} \binom{k + \frac{1}{2}}{l} \binom{-j - k - \frac{1}{2}}{n + k - j - l + 1} (-1)^{n-j} 3^{2j-k-\frac{1}{2}} \frac{\epsilon^j \left(1 + \epsilon + \epsilon^2\right)^{j-2k+1} (1 - \epsilon)^{3l} (1 + \epsilon)^l}{(1 + 2\epsilon)^{3\left(j - k + l - \frac{1}{2}\right)}} $$

$$ D_n = -\frac{2(1 + 2\epsilon)}{3} \sum_{j=0}^{\infty} \sum_{k=0}^{j/2} \sum_{l=0}^{k} \binom{\frac{1}{3}}{j} \binom{j}{2k} \binom{k}{l} \binom{j - k}{n - k - l} (-1)^{j+n} 3^{2j-k} \frac{(1 + \epsilon)^{j+l} (1 - \epsilon)^{3l} (1 + 2\epsilon)^{3(k-l)} \left(1 + \epsilon + \epsilon^2\right)^{j-2k}}{(2 + \epsilon)^{3j}} $$

$$ D_\nu = -\frac{2(1 + 2\epsilon)}{3} \sum_{j=0}^{\infty} \sum_{k=0}^{j/2} \sum_{l=0}^{\infty} \binom{\frac{1}{3}}{j} \binom{j}{2k + 1} \binom{k + \frac{1}{2}}{l} \binom{j - k - \frac{1}{2}}{n + k - l + 1} (-1)^{n+j} 3^{2j-k-\frac{1}{2}} \frac{(1 + \epsilon)^{j+l} (1 - \epsilon)^{3l} (1 + 2\epsilon)^{3\left(k-l+\frac{1}{2}\right)} \left(1 + \epsilon + \epsilon^2\right)^{j-2k-1}}{(2 + \epsilon)^{3j}} $$

$$ \nu = n + \frac{1}{2} $$

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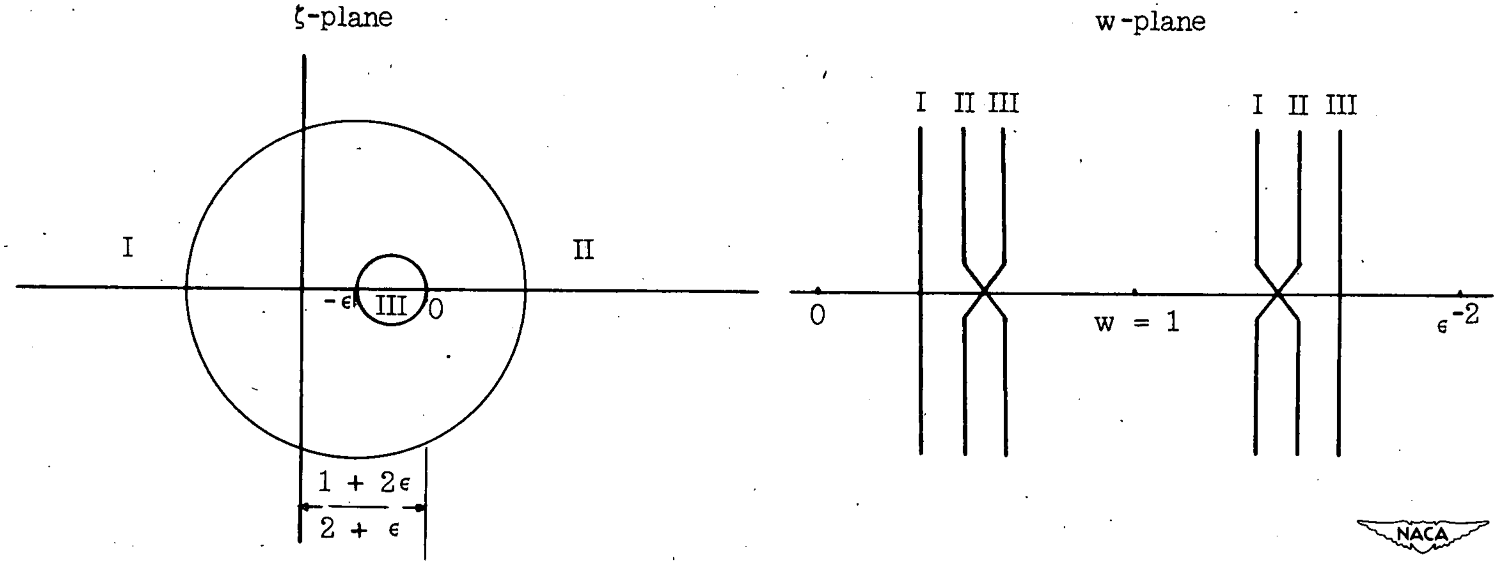

Figure 1.- Conformal representation of incompressible flow about symmetric Joukowski airfoil without circulation.

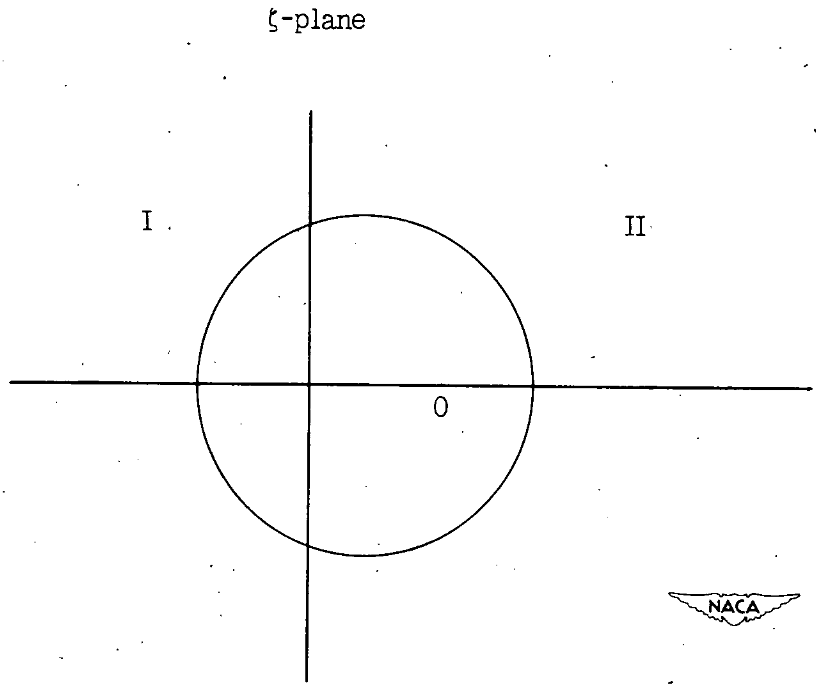

Figure 2.- Simplified ζ-plane for small values of ε.

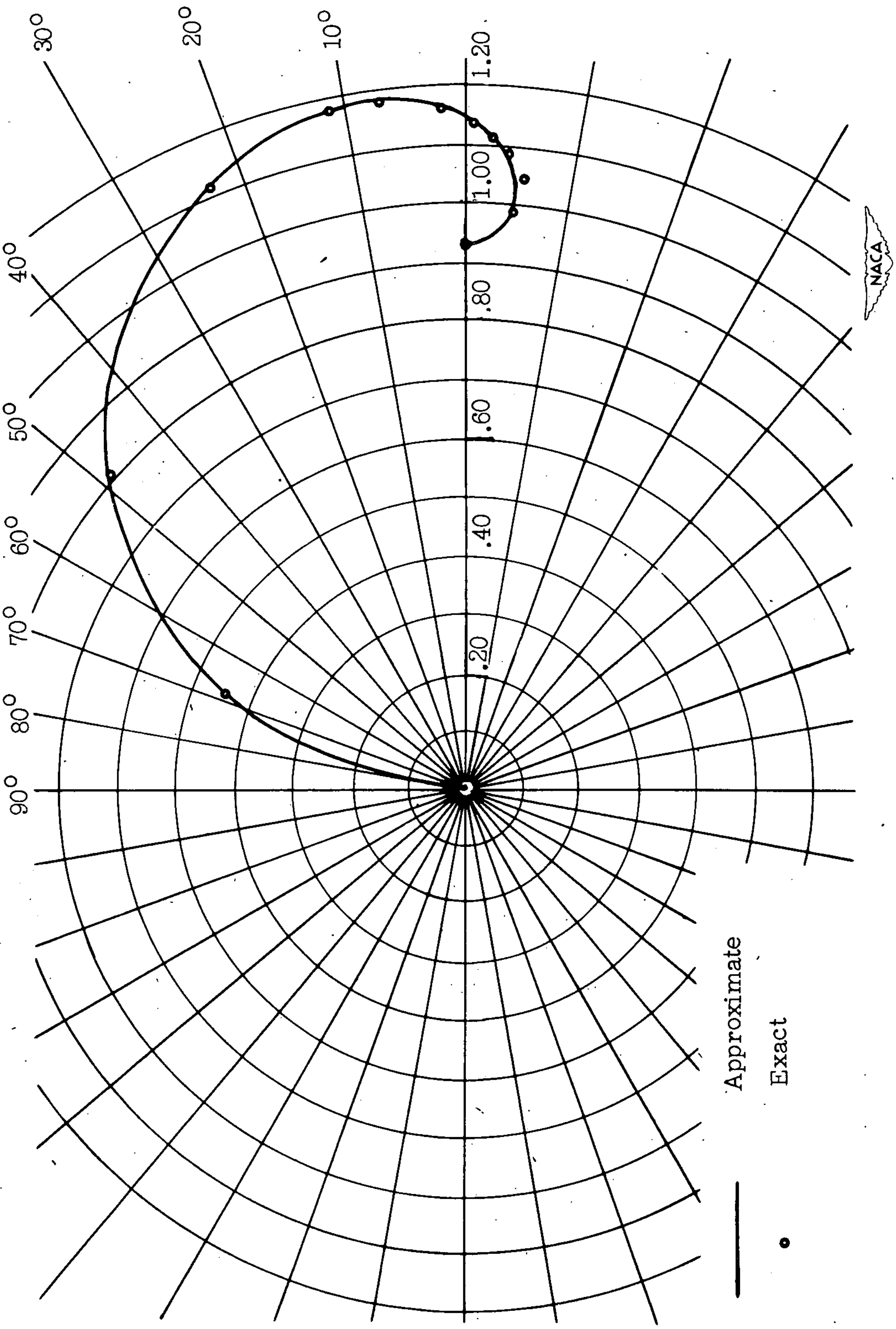

Figure 3.- Zero streamline of incompressible flow around symmetric airfoil in hodograph plane. $\alpha = 0$; $\epsilon = 0.08$.

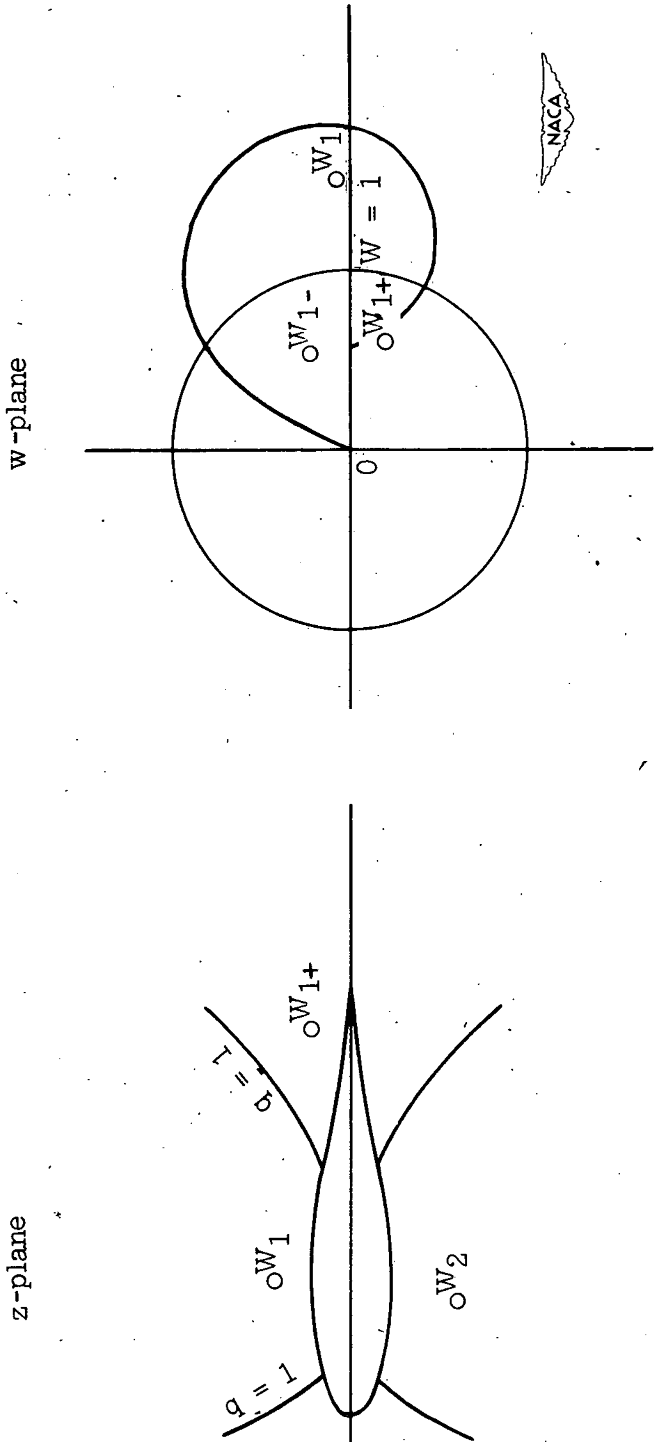

Figure 4.- Diagram showing correspondence between domains of z-plane and w-plane.

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